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GHSS iterative method for image restoration

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***Abstract*—In this study, we introduce the special Hermitian and skew-Hermitian splitting (SHSS) iterative method. The generalized version of HSS (GHSS) iterative method is also introduced to use in image restoration problem. Moreover, we present a new splitting for coefficient matrix in image restoration problem to apply in the GHSS method. Convergence of the proposed method is also investigated. Finally, three numerical examples are given to illustrate the efficiency and accuracy of the GHSS iterative method.**

Keywords*—*image restoration; Hermitian and skew-Hermitian splitting iterative method; generalized Hermitian and skew-Hermitian splitting iterative method; boundary conditions

I. Introduction

In image restoration process, a priori knowledge of the degradation phenomenon is used to reconstruct or recover the degraded image. Image restoration is a major problem in many fields of applied sciences such as medical and astronomical imaging, engineering, optical systems and many other areas [4], [7]. Several methods have been applied to investigate the solution of this problem such as wavelet transform method [4], total variation (TV) method [3] and least square method [6]. In this paper, we consider the matrix-vector form of the image restoration problem which can be written as [4]:

$$g = Af + \eta, \tag{1}$$

where A is a blurring matrix of size $n^2 \times n^2$ and f, η, and g are n^2-dimensional vectors representing the original image, additive noise, and degraded image, respectively. Note that we use the point spread function (PSF) to construct the blurring matrix A. Since the observed image g is of finite dimension and Eq. (1) is obtained from a convolution process, we consider some boundary conditions (BCs) on the f outside the observed image domain. The structures of the matrix A is depend on these BCs.

In this work, we consider zero, periodic and reflexive BCs to restore images [5]. In zero BCs, the outside boundary of the exact image is assumed to be black. In this case, the blurring matrix A is a block Toeplitz with Toeplitz blocks (BTTB) matrix. Periodically extension of the data outside the domain of consideration leads to periodic BCs. For this case, the matrix A has block circulant with circulant blocks (BCCB) structure. In zero and periodic BCs, Fast Fourier Transforms (FFTs) are effectively applied to matrix-vector multiplications. In reflexive BCs, the original scene inside the boundary immediately reflects itself outside of the boundary. In this case, the matrix A is block Toeplitz-plus-Hankel with Toeplitz-plus-Hankel blocks (BTHTHB). Moreover, for symmetric PSFs, the two-dimensional fast cosine transform (DCT) can be used to diagonalize the matrix A.

It is noted that the linear system (1) is ill-conditioned. The Tikhonov method is one of the most known methods for solving ill-conditioned linear systems [5]. In the proposed method, the linear system (1) is substituted by solving the following problem:

$$\min_f \|Af - g\|_2^2 + \mu^2 \|Lf\|_2^2, \tag{2}$$

where L is a regularization matrix and $\mu > 0$ is called the regularization parameter. Note that the balance between the minimization of $\|Af - g\|_2^2$ and the regularization term $\|Lf\|_2^2$ is controlled by regularization parameter. In this study, we consider Eq. (2) when $0 < \mu < 1$ and $L = I$. Hansen et al. have shown that the Tikhonov minimization problem is mathematically equivalent to solving the following equation [5]:

$$(A^T A + \mu^2 I)f = A^T g. \tag{3}$$

To find the solution of the system (3), Lv et al. presented the following equivalent system [8]:

$$\underbrace{\begin{bmatrix} I & A \\ -A^T & \mu^2 I \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} e \\ f \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} g \\ 0 \end{bmatrix}}_{b}, \tag{4}$$

where K is $2n^2 \times 2n^2$ non-Hermitian positive definite matrix, I is the $n^2 \times n^2$ identity matrix and the additive noise is represented by auxiliary variable e, i.e $e = g - Af$.

The Hermitian and skew-Hermitian splitting (HSS) iterative method has been presented to solve non-Hermitian positive definite linear systems [1]. A generalization of the HSS (GHSS) iterative method has been presented by Benzi [2] to solve a class of non-Hermitian linear systems [2]. In the GHSS method, the Hermitian part of the coefficient matrix of linear system is splitted to positive definite and positive semi-definite matrices. Lv et al. used the idea of the HSS iterative method and presented a special case of the HSS (SHSS) method to solve the image restoration problem (4).

In this paper, we present a new splitting to the Hermitian part of coefficient matrix in (4) to use in the GHSS method. Convergence of the proposed method for image restoration problem (4) is also investigated.

This paper is organized as follows. In Section 2, we give a description of the SHSS and the GHSS methods to solve linear systems and image restoration problem. We present a new splitting of iteration matrix for the application in GHSS method. In Section 3, three examples are given to show the effectiveness and accuracy of the proposed method. Finally, some concluding remarks are presented in Section 4.

II. Description of the method

Suppose that A be a non-Hermitian matrix. To implement the HSS method, we first split the matrix A as

$$A = H + S, \tag{5}$$

where

$$H = \frac{1}{2}(A + A^T), \qquad S = \frac{1}{2}(A - A^T). \tag{6}$$

Then, the two splittings of A are presented as:

$$A = (H + \alpha I) - (\alpha I - S), \quad A = (S + \alpha I) - (\alpha I - H),$$

where α is a positive parameter. By alternating between the two splittings, the HSS iterative method for solving the proposed system is given for $k = 0, 1, \ldots$ as

$$\begin{cases} (H + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\ (S + \alpha I)x_{k+1} = (\alpha I - H)x_{k+\frac{1}{2}} + b, \end{cases} \tag{7}$$

where x_0 is a given initial guess. Lv et al. [8] presented SHSS method by substituting $\alpha := 1$ in the second equation of Eq. (7). In other word, the SHSS method can be written as a following two-step iterative method for $k = 0, 1, \ldots$:

$$\begin{cases} (H + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - S)x_k + b, \\ (S + I)x_{k+1} = (I - H)x_{k+\frac{1}{2}} + b. \end{cases} \tag{8}$$

Note that Hermitian and skew-Hermitian parts for the image restoration problem presented in Eq. (4) are given as follows:

$$\begin{aligned} K &= \begin{bmatrix} I & A \\ -A^T & \mu^2 I \end{bmatrix} \\ &= \begin{bmatrix} I & O \\ O & \mu^2 I \end{bmatrix} + \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = H + S. \end{aligned} \tag{9}$$

where O is the $n^2 \times n^2$ zero matrix.

Remark 1. ([8]) If σ_1 and σ_n be the largest and smallest singular values of the matrix A, respectively, then for $\alpha > \frac{\sigma_1^2 - \mu^2 - 2\mu^2\sigma_1^2}{2 - \mu^2 + \sigma_1^2}$ the SHSS iteration method is convergent for any initial vector x_0. Moreover, the optimal value of α in the proposed method for image restoration problem is given by:

$$\alpha^* = \frac{\sigma_1^2 + \sigma_n^2 + 2\sigma_1^2\sigma_n^2}{2 + \sigma_1^2 + \sigma_n^2}. \tag{10}$$

Note that for the optimal value α^*, we have the most convergence speed of the SHHS method. It has been shown that the SHSS method is more effective than the HSS method for image restoration [8].

Benzi presented the GHSS method to solve the linear systems with non-dominant Hermitian part and skew-Hermitian part of the coefficient matrix [2]. In the GHSS method the Hermitian part of H is decomposed as

$$H = G + P = \epsilon L + P, \tag{11}$$

where L is a Hermitian positive definite, P is a Hermitian positive semidefinite and $\epsilon > 0$ is a small constant. The GHSS iteration is given as follows for $k = 0, 1, \ldots$:

$$\begin{cases} (G + \alpha I)x_{k+\frac{1}{2}} = (\alpha I - P - S)x_k + b, \\ (S + P + \alpha I)x_{k+1} = (\alpha I - G)x_{k+\frac{1}{2}} + b, \end{cases} \tag{12}$$

where x_0 is an initial guess. Now, we introduce the following lemma for the convergence of the proposed GHSS iterative method to solve linear systems.

Lemma 1. ([2]) Suppose that the matrix A is splitted as $A = H + S = (G + P) + S$, where G and P are Hermitian positive semidefinite and S is skew-Hermitian. If either G or K is positive definite, alternating iteration (12) converges unconditionally to the unique solution of Ax = b.

Now, to implement the GHSS method in image restoration problem, a new splitting for the Hermitian part of K is presented. This splitting is given as follows:

$$\begin{aligned} K = &\begin{bmatrix} \frac{1}{2}I & O \\ O & \beta\mu^2 I \end{bmatrix} + \begin{bmatrix} \frac{1}{2}I & O \\ O & (1-\beta)\mu^2 I \end{bmatrix} \\ &+ \begin{bmatrix} O & A \\ -A^T & O \end{bmatrix} = G + P + S, \end{aligned} \tag{13}$$

where β is a positive constant. By using the splitting (13), we can implement the GHSS iteration (12) to solve image restoration problem (4). Note that a simple matrix-vector multiplication can be used to solve the first equation in (12):

$$\begin{aligned} x_{k+\frac{1}{2}} = &\begin{bmatrix} \frac{1}{0.5+\alpha} I & O \\ O & \frac{1}{\alpha+\beta\mu^2} I \end{bmatrix} \\ &\times \left(\begin{bmatrix} (\alpha - 0.5)I & -A \\ A^T & (\alpha + (\beta - 1)\mu^2)I \end{bmatrix} x_k + b \right). \end{aligned} \tag{14}$$

Since the Krylov subspace methods are effective on problems containing matrix-vector multiplications. We use the GMRES method [9] to solve the second equation in GHSS iteration (12). After some simple manipulations of proposed algorithm in [8], we can implement GHSS method as the Algorithm 1.

Remark 2. In the third step of Algorithm 1, we use $e_{k+1}^{(0)} = e_{k+1/2}$ and $f_{k+1}^{(0)} = f_{k+1/2}$ as the initial guesses for GMRES method. Moreover, consider the residual of the proposed method as:

$$q_j = \begin{bmatrix} (\alpha - 0.5)e_{k+\frac{1}{2}} + g - (\alpha + 0.5)e_{k+1}^{(j)} - Af_{k+1}^{(j)} \\ (\alpha - \beta\mu^2)f_{k+\frac{1}{2}} + A^T e_{k+1}^{(j)} - (\alpha + (1-\beta)\mu^2)f_{k+1}^{(j)} \end{bmatrix}.$$

The GMRES method acts until $\|q_j\|_2 / \|q_0\|_2 < \zeta$, where ζ is a very small positive value.

In the next theorem, we show that proposed GHSS method with our splitting is convergent for image restoration problem.

Theorem 1. *Assume that $0 < \beta \leq 1$ and the matrices G, P and S are defined as Eq. (13). Then the iteration (12) unconditionally converges to the unique solution of $Kx = b$.*

Algorithm 1: The GHSS iterative method

1. Choose the initial guess of original image $f_0 = g$, initial value of noise $e_0 = g - Af_0$, maximum number of outer iteration M and very small positive τ ;
2. $r_0 := b - Kx_0$;
3. **for** $k = 0, 1, 2, \dots,$ *until* $\frac{\|r_k\|_2}{\|r_0\|_2} > \tau$ *or* $k < M$ **do**

$$e_{k+\frac{1}{2}} := \frac{1}{0.5+\alpha}((\alpha+0.5)e_k - Af_k + g),$$

$$f_{k+\frac{1}{2}} := \frac{1}{\alpha+\beta\mu^2}(A^T e_k + (\alpha+(\beta-1)\mu^2)f_k),$$

$$\text{Solve}\begin{cases}(\alpha+0.5)e_{k+1} + Af_{k+1} = (\alpha-0.5)e_{k+\frac{1}{2}} + g,\\ -A^T e_{k+1} + (\alpha+(1-\beta)\mu^2)f_{k+1}\\ \qquad = (\alpha-\beta\mu^2)f_{k+\frac{1}{2}}.\end{cases}$$

$$r_{k+1} := b - Kx_{k+1},$$

end

Proof: For $0 < \mu, \beta \leq 1$, it can be easily seen that G is Hermitian positive definite and P is positive semidefinite matrix. Moreover, S is skew-Hermitian part of splitting. Hence from lemma 1, the proposed method unconditionally converges to the unique solution of $Kx = b$. ■

III. Numerical Examples

In this section, we consider three examples to show the efficiency and accuracy of the proposed method. All examples are implemented in Matlab 8.1 software.

The peak signal-to-noise ratio (PSNR) is defined as follows to compare the original image with the restored one:

$$\text{PSNR} = 10\log_{10}\frac{4\times 255^2\times n^4}{\|f - f_{\text{true}}\|_2^2}$$

where f_{true} and f are true and restored images, respectively. Furthermore, the relative error is given by $\|f_{\text{true}} - f\|_2/\|f_{\text{true}}\|_2$. In all examples, the optimal values of α and β are approximately estimated by some tests on an image in the GHSS method. The optimal values of α in the SHSS method, is obtained by the Eq (10). The restarted GMRES method of MATLAB with $restart = 15$ and $\zeta = 10^{-6}$ has been used to solve the proposed system in Algorithm 1. Furthermore, the stopping tolerance of the proposed algorithm is $\tau = 10^{-6}$ and the maximum number of outer iteration is $M = 15$.

Example 1. In this example, the 256×256 cameraman grayscale image is used to investigate the proposed methods. The symmetric truncated Gaussian PSF is used to blur the true image. Moreover, 1% Gaussian white noise is added to blurred image to give degraded image. Finally, the observed image domain is shown by white lines. The proposed PSF function is considered as:

$$h_{ij} = \begin{cases} ce^{-0.1(i^2+j^2)}, & \text{if } |i-j| \leq 8,\\ 0, & \text{otherwise,}\end{cases}$$

where c is a normalization parameter. The true image and degraded image are shown in Fig. 1. The PSNR value of degraded image in this example is 26.06.

TABLE I. Values of (α, β) in Example 1

Method/BC	Zero	Periodic	Reflexive
SHSS	$(0.3333, -)$	(0.3333,-)	$(0.3255, -)$
GHSS	$(0.9, 0.95)$	$(2.1, 0.98)$	$(0.06, 0.1)$

TABLE II. PSNR values for various methods in Example 1

Method/BC	Zero	Periodic	Reflexive
SHSS	21.50	24.29	28.06
GHSS	22.55	26.22	28.51

TABLE III. Relative error of the SHSS and GHSS methods for Example 1

Method/BC	Zero	Periodic	Reflexive
SHSS	0.3296	0.2389	0.1547
GHSS	0.2916	0.1911	0.1470

Fig. 1. True (left) and degraded image (right) in Example 1

Fig. 2. Restored image with GHSS method for zero (left) periodic (middle) and reflexive (right) BCs in Example 1

The chosen values of (α, β) for SHSS and GHSS methods in this example are given in Table I. The PSNR of restored images with proposed methods are available in Table II. Moreover, the relative error of the SHHS and the GHSS methods is given in Table III. The restored images by using the GHSS method for zero, periodic and reflexive BCs are shown in Fig. 2. As the results show, the GHSS method is more accurate and effective than SHSS method.

Example 2. In this example, we consider the 128×128 simulated MRI of a human brain which is available in the MATLAB Image Processing Toolbox. To blur the image, the introduced out-of-focus PSF function in [5] is used with $dim = 9$ and $R = 4$. The degraded image is obtained by adding 2% Gaussian noise to blurred image.

The true and degraded images are shown in Fig. 3. In this example, the PSNR value of the degraded image is 32.39. The chosen values of (α, β) are given in Table IV. The PSNR values of restored images are presented in Table V. Moreover, the relative error of the SHHS and the GHSS methods is given in Table VI. As the numerical results show, the GHSS method is more accurate than SHSS method to restore images. For more investigation, the restored images with GHSS method for zero, periodic and reflexive BCs are shown in the Fig. 4.

TABLE IV. VALUES OF (α, β) IN EXAMPLE 2

Method/BC	Zero	Periodic	Reflexive
SHSS	$(0.3377, -)$	$(0.3333, -)$	$(0.3255, -)$
GHSS	$(0.11, 0.18)$	$(0.13, 0.25)$	$(0.21, 0.3)$

TABLE V. PSNR VALUES FOR VARIOUS METHODS IN EXAMPLE 2

Method/BC	Zero	Periodic	Reflexive
SHSS	34.99	35.14	35.13
GHSS	35.76	35.92	35.68

TABLE VI. RELATIVE ERROR OF THE SHSS AND GHSS METHODS FOR EXAMPLE 2

Method/BC	Zero	Periodic	Reflexive
SHSS	0.2340	0.2302	0.2304
GHSS	0.2147	0.2103	0.2162

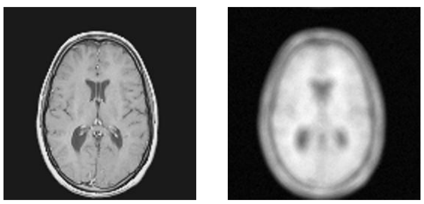

Fig. 3. True (left) and degraded image (right) in Example 2

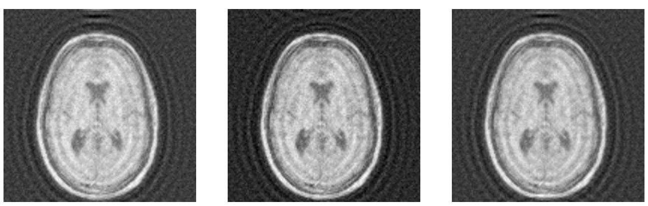

Fig. 4. Restored image with GHSS method for zero (left) periodic (middle) and reflexive (right) BCs in Example 2

Example 3. In this example, we consider an astronomical image from a ground-based telescope. The proposed image is blurred by using Keck telescope PSF. Degraded image is obtained by adding 6% Gaussian noise to the blurred image. The true image and degraded image are given in Fig. 5. Now, we consider the SHHS and GHSS methods and compare them for periodic BCs. The number of outer iteration in both methods is supposed as $M = 25$. In Fig. 6, the restored images with SHSS and GHSS methods are shown for $\alpha = 0.3383$ and $(\alpha, \beta) = (0.1, 0.3)$, respectively.

The residual error in k-th outer iteration of Algorithm 1 is computed by $e_k = g - Af_k$. In Fig. 7, $\|e_k\|_2/\|e_0\|_2$ versus the outer production number k is plotted to consider the convergence speed of proposed methods for periodic BCs. As we can see, the convergence speed of the GHSS method is more faster than the SHSS method.

IV. CONCLUSION

In this paper, image restoration problem was reduced to solve a non-Hermitian positive definite linear system. A new splitting is presented for the coefficient matrix to use in GHSS method. The numerical results were also given. As the numerical results show, our method is accurate and effective in image restoration problem.

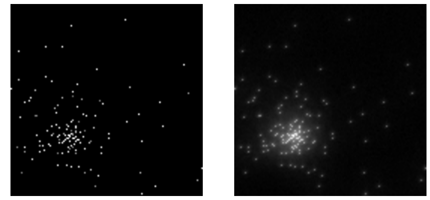

Fig. 5. True image (left) and degraded image (right) in Example 3

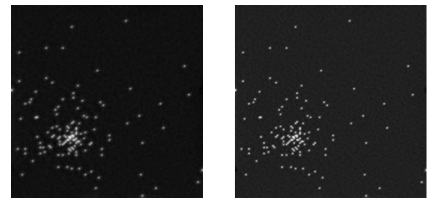

Fig. 6. Restored image with SHHS (left) and GHSS (right) methods in Example 3

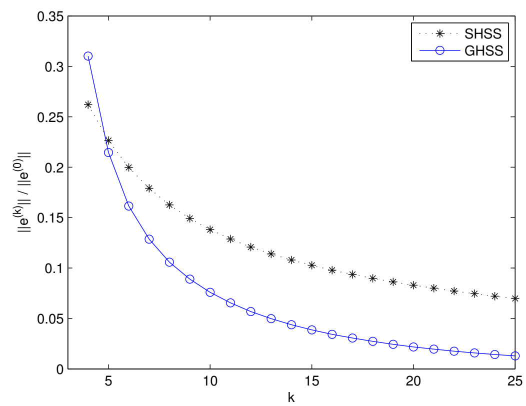

Fig. 7. Comparison between residual errors for periodic BC in Example 3

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