One step ahead Nonlinear Predictive Control of Cart with Pendulum with Input Constraints

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***Abstract*—This article was intended to present the contributions and fundamental components of a tool for predictive control of nonlinear systems. Indeed the study has been devoted to the one step ahead predictive control law. The importance of this type of control is its effect by operation of the feedforward path to follow in the future, on the other hand, it is possible to exploit fully the information of predefined trajectories, the purpose of this strategy is matching the output of the process with setpoint in the future. By cons most conventional control laws will not bring into account the future behavior of the command to the present. In this work a fixed step on head predictive control with input constraints has been applied to a nonlinear system, that of a cart with pendulum in order to control cart position while maintaining the angle of pendulum to the equilibrium position in presence of input constraints (force amplitude)**

Keywords*— *one step ahead predictive control, cost fonction, pendulum'angle, nonlinear system, input constraints.

I. Introduction

The concept of predictive control is the creation of an anticipatory effect, this control structure, developed for linear systems, has experienced a real boom as advanced control technology since the 80s [1]. This growth is due to its robustness vis-à-vis the structured or unstructured uncertainties. In general, the dynamic model of physical processes is nonlinear and the establishment of predictive control laws for these processes requires minimizing the cost function online, which is an operation very complex [2]. To avoid this problem of online optimization, nonlinear predictive control several off-line have been proposed [3] [4][5].

The prediction of tracking discard at one step is obtained using Taylor expansion of order r_i of the output signal and reference, where r_i is the relative degree of the i^{th} system output, the solution of the minimization a quadratic criterion at one step establishes the control law.

In this paper a fixed one step ahead nonlinear predictive control with input constraints is applied to the cart with pendulum system. The choice of such systems is motivated by the complexity of their dynamic behaviors. These systems are often used in research laboratories to validate the control laws developed theoretically [1]-[2].

The complexity of this system lies in the fact that it is a nonlinear system, unstable open loop and under powered **,** meaning that with a single input we should control two output variables.

After presenting the principle of a fixed one step ahead predictive control with input constraints, a mathematical model is developed for our nonlinear system to validate and test the proposed control, simulations were conducted through which the control performance is evaluated

II. Nonlinear Predictive Control

Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i(t) \\ y(t) = h(x) \end{cases} \qquad (1)$$

Where x (t) is the vector of state variables, u (t) is the control vector and y (t) is the output vector, the functions f, g, h are assumed to be real and have continuous partial derivatives .

The classical goal in control is to impose the output of the controlled system to achieve a setpoint as quickly as possible [6]. In the predictive context, the predicted tracking error is minimized over a finite horizon. The model prediction of a nonlinear system is a continuous function that allows us to calculate the system output at future time (t + h), where h> 0 is the prediction horizon.

The predictive model output based on the Taylor series expansion is given by,

$$y(t+h) = y(t) + V_y(x,h) + \Lambda(h)W(x)u \qquad (2)$$

Where

$$V_y(x,h) = (v_1(x,h) \quad v_2(x,h) \quad \ldots. \quad v_m(x,h))^T$$

With

$$v_i(x,h) = hL_f h_i(x) + \frac{h^2}{2!} L_f^2 h_i(x) + \cdots \qquad + \frac{h^{r_i}}{r_i!} L_f^{r_i} h_i(x)$$

$$\Lambda(h) = diag\left(\frac{h^{r_1}}{r_1!}, \frac{h^{r_2}}{r_2!}, \dots, \frac{h^{r_m}}{r_m!}\right)$$

$$W(x) = (w_1 \quad w_2 \quad \cdots \quad w_m)^T$$

With

$$w_i(x) = \left(L_{g1} L_f^{r_i - 1} h_i(x) \quad \dots \quad L_{gm} L_f^{r_i - 1} h_i(x)\right)$$

A. Reference Trajectory

For the output y (t) of nonlinear system (1) can follow the reference trajectory y_{ref} (t), it must be r differentiable, where r is the relative degree of the output y (t). This condition ensures the controllability of the output along the setpoint y_{ref} (t) [7].

Therefore we can apply the Taylor expansion of order r to the reference signal:

$$y_{ref}(t+h) = y_{ref}(t) + d(t,h) \qquad (3)$$

Where

$$d(t,h) = (d_1(t,h) \quad d_2(t,h) \quad \dots \quad d_m(t,h))^T$$

With

$$d_i(t,h) = h\dot{y}_{refi} + \frac{h^2}{2!}\ddot{y}_{refi} + \dots + \frac{h^{r_i}}{r_i} y_{refi}^{(r_i)}$$

In case this is not checked, a trajectory model of exponential type is used to generate the reference trajectory yref (t) from the setpoint yd (t) [5]. The reference trajectory yref (t) is in this case the solution of differential equation:

$$y_{ref}(t) + \gamma_1 \frac{dy_{ref}}{dt} + \gamma_2 \frac{dy_{ref}}{dt} \dots + \gamma_r \frac{dy_{ref}}{dt} = y_d \quad (4)$$

B. One Step Ahead Predictive Control

The objective of one step ahead predictive control nonlinear is to find a control law u (t) which coincides the output y (t) with the reference trajectory y_{ref} (t) at time (t + h) [3]. So the criterion is to minimize the following functional:

$$J_1(y, y_{ref}, R, Q, u) = \frac{1}{2}\left\| y(t+h) - y_{ref}(t+T)\right\|_Q^2 + \frac{1}{2}\|u(t)\|_R^2 \quad (5)$$

Where Q ϵ $\mathbf{R}^{m\times m}$ is a definite positive matrix and R ϵ $\mathbf{R}^{m\times m}$ is a positive semi definite matrix.

The optimal solution is then obtained by minimizing the criterion (5) for the nonlinear system (1) compared to control vector u (t)

$$u(t) = [(\Lambda W)^T Q \Lambda W + R]^{-1} (\Lambda W)^T Q\left[e(t) + V_y(x,h) - d(t,x)\right] \qquad (6)$$

Where e (t) is the tracking error

$$e(t+h) = y(t+h) - y_{ref}(t+h) \qquad (7)$$

C. One Step Ahead Predictive Control with Input Constraints

The cost function given by (5), can be written in quadratic form expressed as

$$J = \frac{1}{2} u^T E u + u^T F \qquad (8)$$

Where

$$E = (\Lambda W)^T Q \Lambda W + R$$

$$F = (\Lambda W)^T Q\left[e(t) + V_y(x,h) - d(t,x)\right]$$

E, *F*: are compatible matrices and vectors in the quadratic programming problem. Without loss of generality, *E* is symmetric and positive definite.

As the optimal solutions will be obtained using quadratic programming, the constraints need to be decomposed into two parts to reflect the lower limit and the upper limit with opposite sign. In case of input constraints which are the most commonly encountered among all constraint types, we demand that

$$U_{min} \le u(k) \le U_{max}$$

Or in a matrix form

$$\begin{bmatrix} -I \\ I \end{bmatrix} u \le \begin{bmatrix} -U_{min} \\ U_{max} \end{bmatrix}$$

With

I: is(m×m) identity matrix

m: Number of inputs

So the objective of one step ahead predictive control, which allow us to find the control law in a way that coincides the output with the desired trajectory and that satisfied the input constraints, can be expressed by the following functional:

$$J = \frac{1}{2} u^T E x + u^T F \qquad (9)$$

$$Dx \le \gamma$$

Where

$$D = \begin{bmatrix} -I \\ I \end{bmatrix};\ \gamma = \begin{bmatrix} -\mathrm{U_{min}} \\ \mathrm{U_{max}} \end{bmatrix}$$

Once the problem of control is translated in a quadratic function (9), a simple quadratic programming algorithm can be used to find the control law; in this work Lagrange method is chosen.

III. MODELING OF CART WITH PENDULUM SYSTEM

The cart with pendulum is a multi-variable nonlinear unstable with fast time constants. It is a system with two degrees of freedom, which are represented by two generalized coordinates, x for the horizontal movement of the cart, θ for the rotation of the pendulum [10] [11].

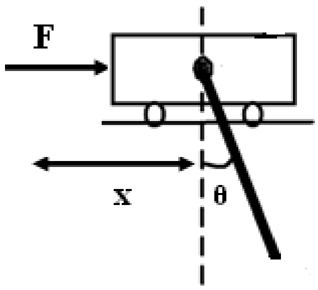

Fig. 1. Diagram of cart with pendulum system

With

m : Pendulum mass

M : Cart mass

l : half length of the pendulum

F(t) : force exerted on the cart

d : friction pendulum

b : friction movement of the cart

x(t) : position of the cart

θ(t) : pendulum's angle

g : intensity of gravity

By applying Newton's second law, we obtain the dynamic equations of the system [11]:

$$\begin{cases} (M+m) + b\dot{x} + mlcos\theta\ddot{\theta} - mlsin\theta\dot{\theta}^2 = F \\ ml\ddot{x}cos\theta + (ml^2+I)\ddot{\theta} + d\dot{\theta} - mglsin\theta = 0 \end{cases}$$

With

$$I = \frac{ml^2}{12}$$

Recall that the state representation for nonlinear systems is as follows:

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i(t) \\ y_i(t) = h_i(x) \end{cases} \quad (7)$$

Suppose the state vector x = (x1 x2 x3 x4)

Where

$x = (\theta \quad \dot{\theta} \quad x \quad \dot{x})^T$ et $u = F$

Matrices of state representation is given by

$$f(x) = \begin{bmatrix} x_2 \\ \dfrac{-(M+m)(mglsinx_1 + dx_2) - mlcosx_1(mlsinx_1{x_2}^2 + bx_4)}{(m+M)(ml^2+I) - (mlcosx_1)^2} \\ x_4 \\ \dfrac{-mlsinx_1\left((ml^2+I){x_2}^2 + mlgcosx_1\right) - mlcosx_1dx_2 - (ml^2+I)bx_4}{(m+M)(ml^2+I) - (mlcosx_1)^2} \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ \dfrac{mlcosx_1}{(m+M)(ml^2+I) - (mlcosx_1)^2} \\ 0 \\ \dfrac{(ml^2+I)}{(m+M)(ml^2+I) - (mlcosx_1)^2} \end{bmatrix}$$

$$h_1 = x_1$$

$$h_2 = x_3$$

IV. APPLICATION OF ONE STEP AHEAD PREDICTIVE CONTROL WITH INPUT CONSTRAINTS TO CART WITH PENDULUM SYSTEM

To illustrate the effectiveness of the nonlinear predictive control with input constraints on our system, cart with pendulum, digital simulations were made on it. Whose goal is to walk the cart along a setpoint trajectory while keeping the pendulum in the equilibrium position and respecting input constraints: Force amplitude, where two cases are chosen: in the first one the force amplitude is limited by 20: U_{min}=-20N, U_{max}=20N, in the second case : U_{min}=-10N, U_{max}=15N

The following values are used to simulate our system:

M=2.4kg; m=0.23kg; l=0.36m; g=9.81m/s²; b=0.05Ns/m; d=0.005Nms/rad;

To evaluate the dynamics of the cart and pendulum, it was considered that the cart must follow a sinusoidal trajectory: $y_{ref}(t)$= 0.5sint and that the pendulum must maintain the equilibrium position of 0 rad.

Using the Taylor expansion of order (r1 = r2 = 2) , we obtain:

$$e(t+h) = e(t) + V_y(x,h) - d(t,h) + \Lambda(h)W(x)u$$

Where

$$\Lambda(h) = \begin{vmatrix} \frac{h^2}{2} & 0 \\ 0 & \frac{h^2}{2} \end{vmatrix} ; \quad W(x) = \begin{vmatrix} L_{g1}L_f h_1 \\ L_{g1}L_f h_2 \end{vmatrix}$$

$$V_y(x,t) = \begin{vmatrix} hL_f h_1 + \frac{h^2}{2} {L_f}^2 h_1 \\ hL_f h_2 + \frac{h^2}{2} {L_f}^2 h_2 \end{vmatrix} \quad ; \; d(t,h) = \begin{vmatrix} h\dot{y}_{ref1} + \frac{h^2}{2} \ddot{y}_{ref1} \\ h\dot{y}_{ref2} + \frac{h^2}{2} \ddot{y}_{ref2} \end{vmatrix}$$

For our command structure simulations were performed to obtain the optimal values of adjustment parameters, including tuning parameters are obtained:

h=0.0008; R=0.000001; Q=[10 0;0 10000];

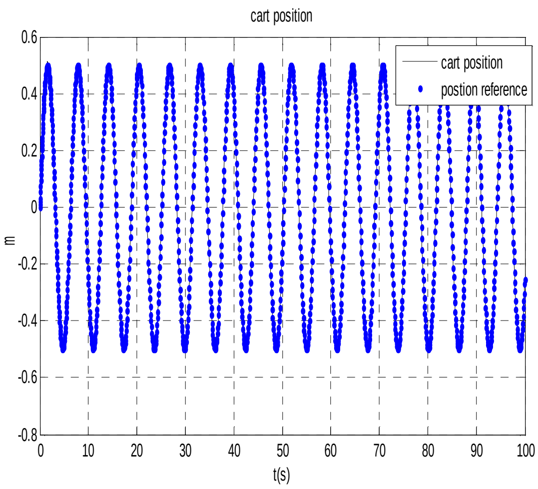

Fig. 2.a. cart position reference and output

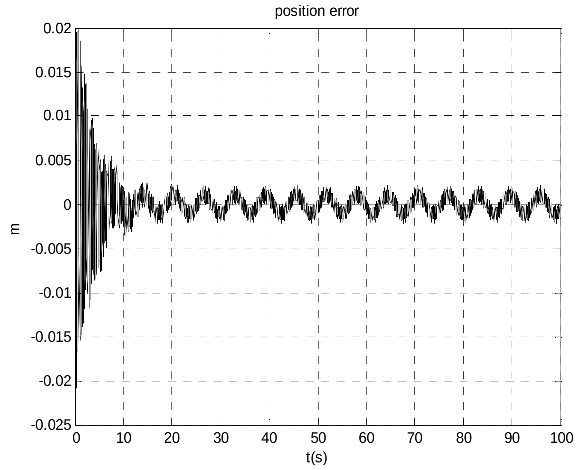

Fig. 2.b. Error between reference and output position

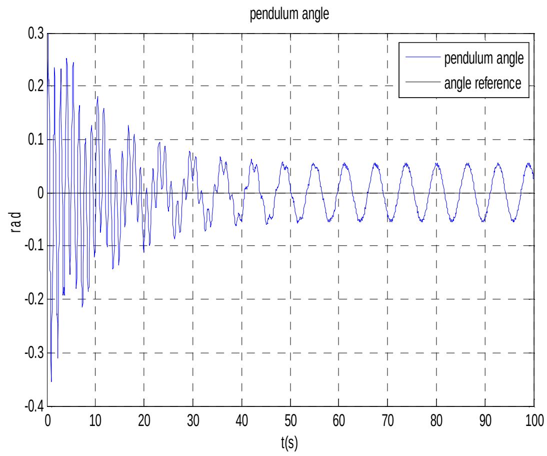

Fig. 2.c. Angle pendulum reference and output

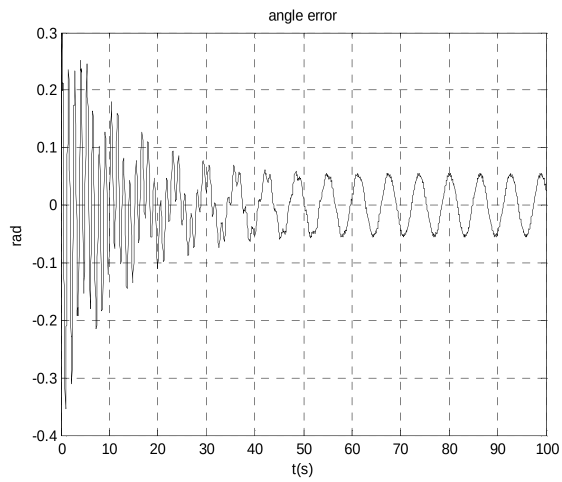

Fig. 2.d. Error between reference and output angle

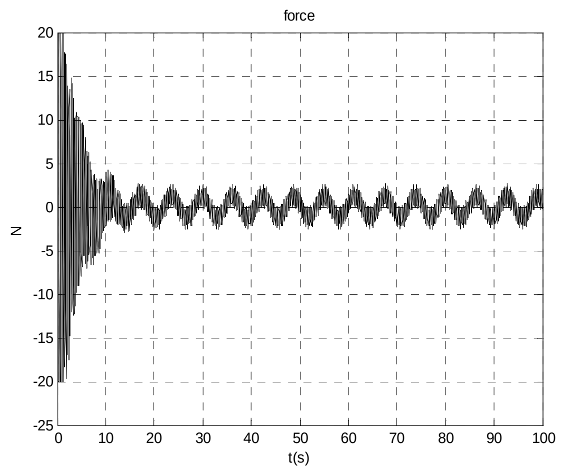

Fig. 2.e. Control (force) applied

Fig. 2. One step ahead predictive control of cart with pendulum system with input constraints U_{max}=20, U_{min}=-20

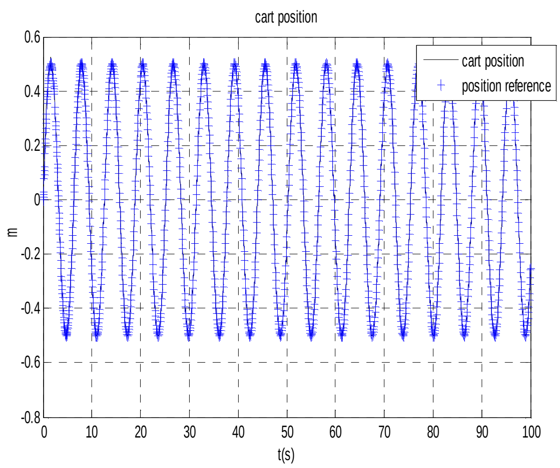

Fig. 3.a. cart position reference and output

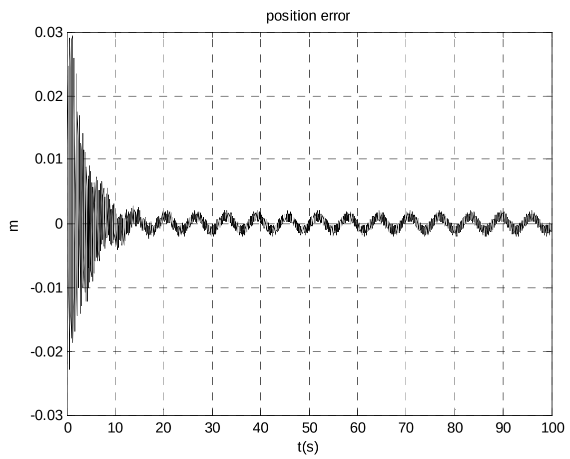

Fig. 3.b. Error between reference and output position

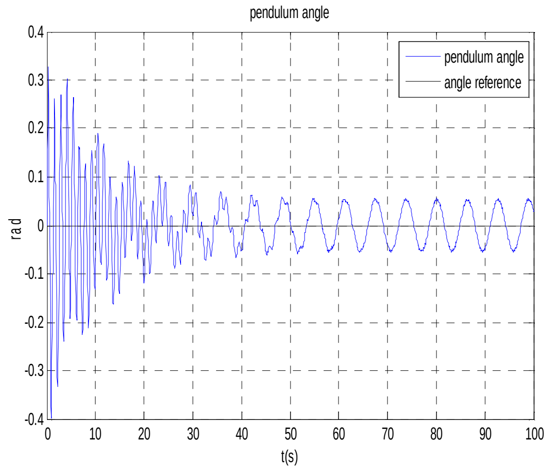

Fig. 3.c. Angle pendulum reference and output

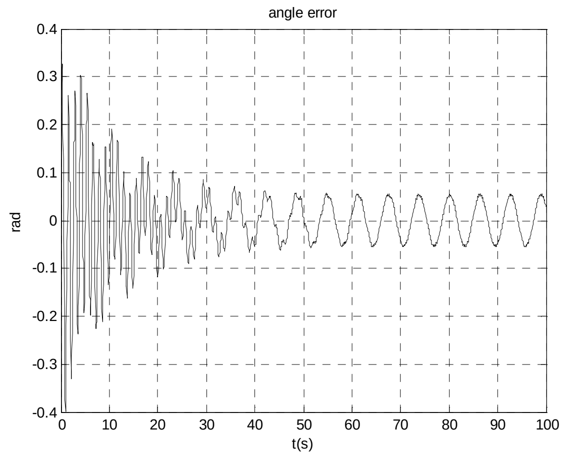

Fig. 3.d. Error between reference and output angle

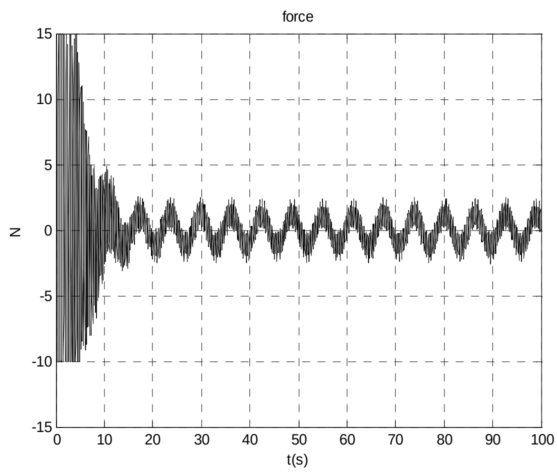

Fig. 3.e. Control (force) applied

Fig. 3. One step ahead predictive control of cart with pendulum system with input constraints U_{max}=15, U_{min}=-10

The control parameters **Q**= [10 0;0 10000],, $\mathbf{R}=10^{-6}\mathbf{I}_n$ and h is set to 0.0008. Simulation results are show in Figures 2 and 3. These Figures give the cart position with reference position, pendulum angle with angle reference position and angle tracking errors also the applied force .

The simulation results clearly show the effectiveness of the one step ahead controller with input constraints in terms of:

- ✓ References tracking (cart position , pendulum angle) where results show a good dynamic tracking cart position (x (t)) and the equilibrium position of pendulum where the angle output takes a very small value which allow it to stay in the equilibrium position , Through these simulation results performed on the nonlinear model of the system, we can see that the one step ahead predictive control law has stabilized the system as well as tracking regulation. This stabilization is achieved by holding the pendulum at its equilibrium position. In spite of various disturbances on the system applied

- ✓ Constraints respecting (Force amplitude) where it's very clear from force figures that the value of these one, in the two cases, still in the interval limited by the minimal and the maximal values.

Finally the major drawback of one step ahead predictive control is the need to perform several simulations to obtain numerical values of optimal parameters settings.

V. Conclusion

This work has focused on the contribution to the development of original control structures, a strategy based on nonlinear predictive mechanism where optimization of the quadratic criterion is used to extract the control law.

One approach has been treated: the one step ahead predictive control with input constraints, whose principle is based on the Taylor series expansion of the predicted output and the reference where the control law is obtained by minimizing the quadratic error between them. Concerning the one step ahead control with input constraints, the basic idea is to translate the constraints problems to linear inequalities then parameterize them using the same parameter vector *u* as the ones used in the design of predictive control, in order to relate to the original model predictive control problem, where the optimal solution is found by solving a quadratic optimization problem.

This approach has been applied to a nonlinear system, that of a cart with pendulum to control the cart position and angle of the pendulum all with respecting input constraints (force amplitude). The simulation results clearly show the effectiveness of this approach in terms of references tracking (cart position, angle pendulum)and respecting input constraints. The control objective is achieved with good accuracy despite the system are unstable. Finally the major drawback of one step ahead predictive control is the need to perform several simulations to obtain numerical values of optimal parameters settings.

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