New Upper Bounds for the Computation of Complementary Error Function

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Abstract—In this article we provide tighter upper bounds for complementary error function which is a monotonically decreasing function widely used in communication theory. The provided bounds approach the exact value in the limiting case. The derivation of the bounds are based on the monotonically increasing property of exponential function used in complementary error function.

Index Terms—Complementary error function, monotonic functions, upper bounds.

I. INTRODUCTION

The error function \( \text{erf}(\cdot) \) or its complementary form \( \text{erfc}(\cdot) \) is very widely used in communication theory. Usually bit error probability or symbol error probability in many communication problems are expressed in terms of either Gaussian \( Q \) function or complementary error function. The numerical values for these functions are tabulated and used for computer simulations. On the other hand, it is sometimes desirable to have an idea about the performance of communication systems without going into detailed computation [1]. Therefore it is important to have closed form expressions or upper bounds for integral functions available in implicit integral form. Explicit type approximations and upper bounds are especially useful for wireless communication systems [2]. In this manuscript we will derive tighter upper bounds for the well known complementary error function \( \text{erfc}(\cdot) \).

The outline of the letter is as follows. In Section II we review the exponential proposed bounds for \( \text{erfc}(\cdot) \). New proposed bounds for \( \text{erfc}(\cdot) \) are explained in Section III. Finally conclusions are drawn in Section IV.

II. BOUNDS FOR COMPLEMENTARY ERROR FUNCTION (ERFC)

The complementary error function is defined as [3]

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \quad (1)
\]

A closely related function to the \( \text{erfc}(\cdot) \) is the \( Q(\cdot) \) function and its relation to the \( \text{erfc}(\cdot) \) is given as

\[
Q(x) = \frac{1}{2} \text{erfc} \left( \frac{x}{\sqrt{2}} \right) \quad (2)
\]

from which the expression of the \( Q(\cdot) \) function can be obtained as

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt. \quad (3)
\]

An alternative form of \( \text{erfc}(\cdot) \) defined as

\[
\text{erfc}(x) = \frac{2}{\pi} \int_0^\frac{\pi}{2} \exp \left( -\frac{x^2}{\sin^2 \theta} \right) d\theta \quad x \geq 0 \quad (4)
\]

is given in [4]. In recent years some upper bounds have been derived for \( \text{erfc}(\cdot) \) and one of them is found as [3]

\[
\text{erfc}(x) \leq 2e^{-x^2} \quad (5)
\]

which is further improved as [3]

\[
\text{erfc}(x) \leq e^{-x^2}. \quad (6)
\]

Later on using the alternative definition of \( \text{erfc}(\cdot) \) in eqn. (4) it was shown in [5] that the bound in eqn. (6) can be obtained by replacing the integrand in eqn. (4) with its maximum at \( \theta = \pi/2 \) as shown below

\[
\text{erfc}(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \exp \left( -\frac{x^2}{\sin^2 \theta} \right) d\theta \quad (7)
\]

\[
\leq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-x^2} d\theta = e^{-x^2}. \quad (8)
\]

In [3] for the derivation of the upper bounds the authors used the monotonically increasing function \( f(\theta) \) whose graph is given in Fig. 1 and defined as

\[
f(\theta) = \exp \left( -\frac{x^2}{\sin^2 \theta} \right) \quad (9)
\]

and using the \( N + 1 \) values of \( \theta \) such that \( 0 = \theta_0 \leq \theta_1 \cdots \leq \theta_N = \pi/2 \) they proposed the following exponential bound

\[
\text{erfc}(x) \leq \frac{2}{\pi} \sum_{n=1}^{N} \int_{\theta_{n-1}}^{\theta_n} \exp \left( -\frac{x^2}{\sin^2 \theta_n} \right) d\theta 
\]

\[
= \sum_{n=1}^{N} a_n \exp(-b_n x^2) \quad (10)
\]

where

\[
a_n = \int_{\theta_{n-1}}^{\theta_n} \exp \left( -\frac{x^2}{\sin^2 \theta} \right) d\theta 
\]

\[
b_n = \frac{1}{\sin^2 \theta_n} \quad (11)
\]

Finally conclusions are drawn in Section IV.
\[ a_n = \frac{2(\theta_n - \theta_{n-1})}{\pi}, \quad b_n = \frac{1}{\sin^2 \theta_n}. \]  

(10)

In bound (9) the right hand side is nothing but numerical evaluation of the integral, and authors used upper Riemann sum of the integral which also provides an upper bound for the \( erf(c) \). With the parameters \( N = 2 \) and \( \theta_1 = \frac{\pi}{4} \) the bound expression in ineq. (9) reduces to

\[ erf(c(x)) \leq \frac{1}{2} e^{-2x^2} + \frac{1}{2} e^{-x^2}. \]  

(11)

Using ineq. (9) different bounds can be derived with different choices of \( N \) and \( \theta_i \).

III. NEW UPPER BOUNDS

In Fig. 1 computer drawn graphs of \( f(\theta) \) for different \( x \) values are depicted. A typical plot of \( f(\theta) \) is shown in Fig. 2 where it is seen that the area under curve of \( f(\theta) \) to the left of \( \theta_c \) may not be accounted in numerical integral computation. In other words, horizontal axis partition can be started from \( \theta_c \) instead of \( \theta = 0 \). Hence, \( N + 1 \) values of \( \theta \) can be chosen as \( \theta_1, \theta_2, \ldots, \theta_N, \theta = \frac{\pi}{2} \) rather than choosing \( \theta_0 = 0 \) as in [3]. This means that we omit the rectangle area to the left of \( \theta_c \) in Riemann sum of the integral. The value of \( \theta_c \) can be determined as follows. Let \( f_{\text{max}} \) denote the maximum value of \( f(\theta) \) which is found as

\[ f_{\text{max}} = \max \{ f(\theta) \} = \exp(-x^2) \quad 0 \leq \theta \leq \frac{\pi}{2}. \]  

(12)

At \( \theta_c \) the amplitude of \( f(\theta) \) is \( K \) times smaller than the maximum value of \( f(\theta) \) such that

\[ f(\theta_c) = \frac{f_{\text{max}}}{K} \]  

(13)

which can be stated as

\[ \exp(-\frac{x^2}{\sin^2 \theta_c}) = \frac{\exp(-x^2)}{K}. \]  

(14)

when solved for \( \theta_c \) we get

\[ \theta_c = \sin^{-1} \left( \pm \sqrt{\frac{x^2}{x^2 + \ln(K)}} \right). \]  

(15)

A variety of upper bounds for different \( N \) values can be derived using eqn. (9) with \( \theta_0 = \theta_c \). For \( N = 1 \),

\[ erf(c(x)) \leq \frac{2}{\pi} \left[ (\frac{\pi}{2} - \theta_c) \times \exp(-x^2) \right] \]  

(16)

which is stated explicitly by substituting \( \theta_c \) expressing in eqn. (15) as

\[ erf(c(x)) \leq \frac{2}{\pi} \left[ (\frac{\pi}{2} - \sin^{-1} \sqrt{\frac{x^2}{x^2 + K}}) \times \exp(-x^2) \right]. \]  

(17)

Choosing \( N = 2, \theta_1 = \frac{\pi}{4} \) and using eqn. (9) we get

\[ erf(c(x)) \leq \frac{2}{\pi} \left[ (\frac{\pi}{4} - \theta_c) \times \exp(-2x^2) + \frac{\pi}{4} \times \exp(-x^2) \right] \]  

(18)

which is written explicitly by substituting \( \theta_c \) expressing in eqn. (15) as

\[ erf(c(x)) \leq \frac{2}{\pi} \left[ (\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{x^2}{x^2 + \ln(K)}}) \times \exp(-2x^2) + \frac{\pi}{4} \times \exp(-x^2) \right]. \]  

(19)

where the right hand side is named as bound-1, and denoted by \( b_1(x) \). In Fig. 2 it is seen that \( \theta_1 \) is the saddle point at which \( f(\theta) \) changes from concave up to concave down. As \( x \) gets larger values \( \theta_i \) shifts towards right, and as \( x \) approaches \( \infty \), \( \theta_i \) approaches \( \pi/2 \). The value of \( \theta_i \) can be determined from the first or second derivatives of \( f(\theta_i) \). The maximum value of \( \frac{d(\theta)}{dx} \) occurs at \( \theta_i \) for a given value of \( x \). This situation is illustrated in Fig. 3 where it is seen that as \( x \) values increase peak of the \( \frac{d(\theta)}{dx} \) shifts towards right.

It is seen in Figs. 3 and 4 that as \( \theta_i \) approaches \( \pi/2 \), \( f(\theta_i) \)
Graphs For Different x Values

Fig. 3. Derivative of the function takes its peak at \(\theta_1\) values mostly becomes concave up, hence trapezoid approximation provides us with an upper bound for \(f(\theta)\). As it is clear from Fig. 4 that using the trapezoid approximation for \(N = 2\), and choosing \(\theta_0 = \theta_2, \theta_1 = \frac{\pi}{4}\), the total area for \(N = 2\) case equals the sum of triangle area \(hfg\) and trapezoid area \(d'fge\) when substituted in eqn. 9 gives the following upper bound

\[
erfc(x) \leq C \times \frac{2}{\pi} \left( \frac{\pi}{4} - \frac{\theta_c}{2} \right) \left( f(\theta'_{c}) + \frac{\exp(-x^2)}{2} \right) \tag{20}\]

where \(\theta'_{c}\) and \(f(\theta'_{c})\) are given as

\[
\theta'_{c} = \frac{\pi}{4} + \frac{\theta_c}{2} \tag{21}
\]

\[
f(\theta'_{c}) = \exp\left(-\frac{x^2}{\sin^2(\theta'_{c})}\right) \tag{22}
\]

and compensating factor \(C\) is added to account for all \(x\) values.

Computer simulation results for the proposed bounds with \(K = 2\) are depicted in Fig. 5 where it is seen that the suggested bound-1 is tighter especially for low \(x\) values than the exponential bound given in eqn. (11). On the other hand bound-2 in eqn. (24) is better especially for large \(x\) values than the exponential bound in eqn. (11). As it is clear from Fig. 5 that the two bound can be combined to yield a better bound. These bounds can be combined into a single bound as using the unit step function \(u(x)\) as:

\[
b(x) = b_1(x) \times (u(x) - u(x - 1)) + b_2(x) \times u(x - 1) \tag{24}
\]
which is written explicitly as follows:

\[
\text{erfc}(x) \leq \frac{2}{\pi} \left( \frac{\pi}{4} - \sin^{-1}\left(\frac{x^2}{\sqrt{x^2 + K}}\right) \right) \\
\times \exp(-2x^2) + \frac{\pi}{4} \times \exp(-x^2) \times (u(x) - u(x - 1)) \\
+ \left[ \frac{\pi}{4} - \frac{1}{2} \times \sin^{-1}\left(\frac{x}{\sqrt{x^2 + \ln(K)}/2}\right) \right] \\
\times \exp\left( -\frac{x^2}{\sin^2\left(\frac{\pi}{4} + \frac{1}{2} \sin^{-1}\frac{x}{\sqrt{x^2 + \ln(K)}/2})\right)} \right) \\
+ \exp\left(-\frac{x^2}{2}\right) \times (u(x - 1)).
\]

(25)

IV. CONCLUSIONS

In this manuscript we proposed new upper bounds for complementary error function. The proposed bounds are tighter when compared to the previously suggested exponential ones. The proposed bound approach the exact values in the limiting case.

REFERENCES


