New Upper Bounds for the Computation of Complementary Error Function

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***Abstract*—In this article we provide tighter upper bounds for complementary error function which is a monotonically decreasing function widely used in communication theory. The provided bounds approach the exact value in the limiting case. The derivation of the bounds are based on the monotonically increasing property of exponential function used in complementary error function.**

***Index Terms*—Complementary error function, monotonic functions, upper bounds.**

I. Introduction

THE error function $erf(\cdot)$ or its complementary form $erfc(\cdot)$ is very widely used in communication theory. Usually bit error probability or symbol error probability in many communication problems are expressed in terms of either Gaussian Q function or complementary error function. The numerical values for these functions are tabulated and used for computer simulations. On the other hand, it is sometimes desirable to have an idea about the performance of communication systems without going into detailed computation [1]. Therefore it is important to have closed form expressions or upper bounds for integral functions available in implicit integral form. Explicit type approximations and upper bounds are especially useful for wireless communication systems [2]. In this manuscript we will derive tighter upper bounds for the well known complementary error function $erfc(\cdot)$.

The outline of the letter is as follows. In Section II we review the exponential proposed bounds for $erfc(.)$. New proposed bounds for $erfc(.)$ are explained in Section III. Finally conclusions are drawn in Section IV.

II. Bounds for Complementary Error Function (Erfc)

The complementary error function is defined as [3]

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{1}$$

A closely related function to the $erfc(\cdot)$ is the $Q(\cdot)$ function and its relation to the $erfc(\cdot)$ is given as

$$Q(x) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{2}}\right) \tag{2}$$

from which the expression of the $Q(\cdot)$ function can be obtained as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt. \tag{3}$$

An alternative form of $erfc(\cdot)$ defined as

$$erfc(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} exp\left(-\frac{x^2}{sin^2\theta}\right) d\theta \quad x \geq 0 \tag{4}$$

is given in [4]. In recent years some upper bounds have been derived for $erfc(\cdot)$ and one of them is found as [3]

$$erfc(x) \leq 2e^{-x^2} \tag{5}$$

which is further improved as [3]

$$erfc(x) \leq e^{-x^2}. \tag{6}$$

Later on using the alternative definition of $erfc(\cdot)$ in eqn. (4) it was shown in [5] that the bound in eqn. (6) can be obtained by replacing the integrand in eqn. (4) with its maximum at $\theta = \pi/2$ as shown below

$$\begin{aligned} erfc(x) &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} exp\left(-\frac{x^2}{sin^2\theta}\right) d\theta \\ &\leq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-x^2} d\theta = e^{-x^2}. \end{aligned} \tag{7}$$

In [3] for the derivation of the upper bounds the authors used the monotonically increasing function $f(\theta)$ whose graph is given in Fig. 1 and defined as

$$f(\theta) = exp\left(-\frac{x^2}{sin^2\theta}\right) \tag{8}$$

and using the $N+1$ values of θ such that $0 = \theta_0 \leq \theta_1 \cdots \leq \theta_N = \pi/2$ they proposed the following exponential bound

$$\begin{aligned} erfc(x) &\leq \frac{2}{\pi} \sum_{n=1}^{N} \int_{\theta_{n-1}}^{\theta_n} exp\left(-\frac{x^2}{sin^2\theta_n}\right) d\theta \\ &= \sum_{n=1}^{N} a_n exp(-b_n x^2) \end{aligned} \tag{9}$$

where

$$a_n = \frac{2(\theta_n - \theta_{n-1})}{\pi}, \;\; b_n = \frac{1}{sin^2\theta_n}. \qquad (10)$$

In bound (9) the right hand side is nothing but numerical evaluation of the integral, and authors used upper Riemann sum of the integral which also provides an upper bound for the $erfc(\cdot)$. With the parameters $N = 2$ and $\theta_1 = \frac{\pi}{4}$ the bound expression in ineq. (9) reduces to

$$erfc(x) \leq \frac{1}{2}e^{-2x^2} + \frac{1}{2}e^{-x^2}. \qquad (11)$$

Using ineq. (9) different bounds can be derived with different choices of N and θ_i.

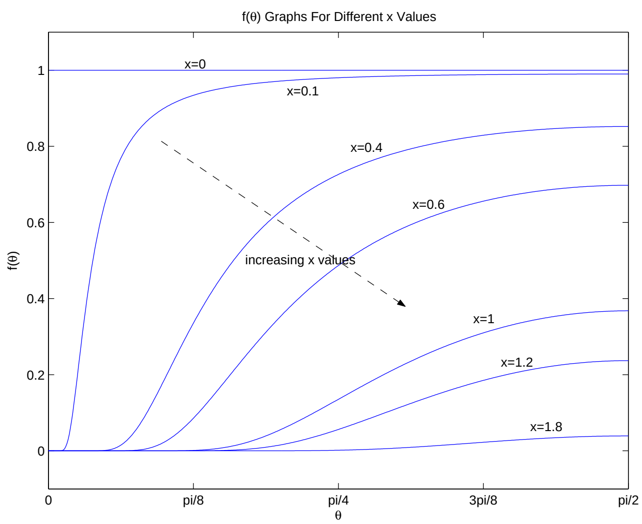

Fig. 1. $f(\theta)$ Curves for different x values

III. New Upper Bounds

In Fig. 1 computer drawn graphs of $f(\theta)$ for different x values are depicted. A typical plot of $f(\theta)$ is shown in Fig. 2 where it is seen that the area under curve of $f(\theta)$ to the left of θ_c may not be accounted in numerical integral computation. In other words, horizontal axis partition can be started from θ_c instead of $\theta = 0$. Hence, $N + 1$ values of θ can be chosen as $\theta_c = \theta_0 \leq \theta_1 \cdots \leq \theta_N = \pi/2$ rather than choosing $\theta_0 = 0$ as in [3]. This means that we omit the rectangle area to the left of θ_c in Riemann sum of the integral. The value of θ_c can be determined as follows. Let f_{max} denote the maximum value of $f(\theta)$ which is found as

$$f_{max} = max\{f(\theta)\} = exp(-x^2); \;\; 0 \leq \theta \leq \frac{\pi}{2} \qquad (12)$$

At θ_c the amplitude of $f(\theta)$ is K times smaller than the maximum value of $f(\theta)$ such that

$$f(\theta_c) = \frac{f_{max}}{K} \qquad (13)$$

which can be stated as

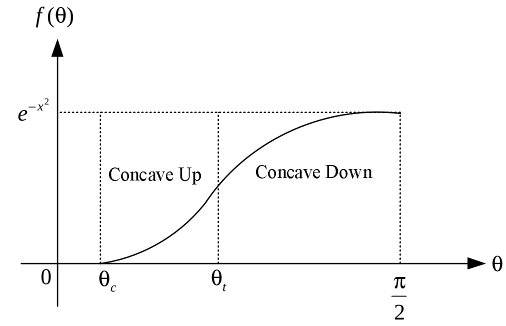

Fig. 2. A Typical Graph of $f(\theta)$ Function

$$exp(-\frac{x^2}{sin^2\theta_c}) = \frac{exp(-x^2)}{K} \qquad (14)$$

when solved for θ_c we get

$$\theta_c = sin^{-1}\left(\pm\sqrt{\frac{x^2}{x^2 + ln(K)}}\right). \qquad (15)$$

A variety of upper bounds for different N values can be derived using eqn. (9) with $\theta_0 = \theta_c$. For $N = 1$,

$$erfc(x) \leq \frac{2}{\pi}\left[(\frac{\pi}{2} - \theta_c) \times exp(-x^2)\right] \qquad (16)$$

which is stated explicitly by substituting θ_c expressing in eqn. (15) as

$$erfc(x) \leq \frac{2}{\pi}\left[(\frac{\pi}{2} - sin^{-1}(\sqrt{\frac{x^2}{x^2 + K}})) \times exp(-x^2)\right]. \qquad (17)$$

Choosing $N = 2$, $\theta_1 = \frac{\pi}{4}$ and using eqn. (9) we get

$$erfc(x) \leq \frac{2}{\pi}\left[(\frac{\pi}{4} - \theta_c) \times exp(-2x^2) + \frac{\pi}{4} \times exp(-x^2)\right] \qquad (18)$$

which is written explicitly by substituting θ_c expressing in eqn. (15) as

$$erfc(x) \leq \frac{2}{\pi}\left[(\frac{\pi}{4} - sin^{-1}(\sqrt{\frac{x^2}{x^2 + ln(K)}})) \times exp(-2x^2) + \frac{\pi}{4} \times exp(-x^2)\right] \qquad (19)$$

where the right hand side is named as bound-1, and denoted by $b_1(x)$. In Fig. 2 it is seen that θ_t is the saddle point at which $f(\theta)$ changes from concave up to concave down. As x gets larger values θ_t shifts towards right, and as x approaches ∞, θ_t approaches $\pi/2$. The value of θ_t can be determined from the first or second derivatives of $f(\theta_t)$. The maximum value of $\frac{df(\theta)}{d\theta}$ occurs at θ_t for a given value of x. This situation is illustrated in Fig. 3 where it is seen that as x values increase peak of the $\frac{df(\theta)}{d\theta}$ shifts towards right.

It is seen in Figs. 3 and 4 that as θ_t approaches $\pi/2$, $f(\theta_t)$

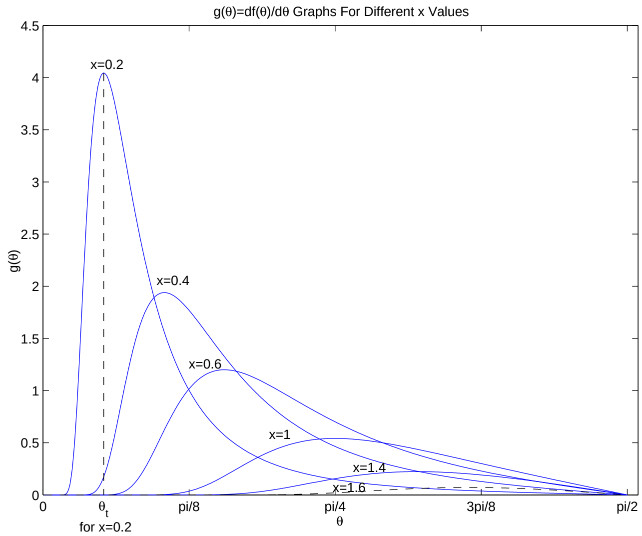

Fig. 3. Derivative of the function takes its peak at θ_t values

mostly becomes concave up, hence trapezoid approximation provides us with an upper bound for $f(\theta)$. As it is clear from Fig. 4 that using the trapezoid approximation for $N = 2$, and choosing $\theta_0 = \theta_c, \theta_1 = \frac{\pi}{4}$, the total are for $N = 2$ case equals the sum of triangle area $\overset{\triangle}{hfg}$ and trapezoid area $'dfge'$ when substituted in eqn. 9 gives the the following upper bound

$$erfc(x) \leq C \times \frac{2}{\pi}\left(\frac{\pi}{4} - \frac{\theta_c}{2}\right)\left(f(\theta_c') + \frac{exp(-x^2)}{2}\right) \quad (20)$$

where θ_c' and $f(\theta_c')$ are given as

$$\theta_c' = \frac{\pi}{4} + \frac{\theta_c}{2} \quad (21)$$

$$f(\theta_c') = exp\left(-\frac{x^2}{sin^2(\theta_c')}\right) \quad (22)$$

and compensating factor C is added to account for all x values.

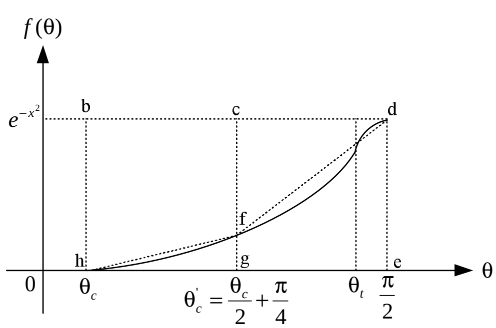

Fig. 4. $f(\theta)$ is mostly concave up for large x values.

The upper bound in ineq. (20) without compensating factor C is valid for large values, to account for small x values also, we added a compensating factor $C > 1$ which is chosen as $C = \pi/2$ via computer simulation, thus the bound in ineq. 20 expands to

$$erfc(x) \leq \left[\frac{\pi}{4} - \frac{1}{2} \times sin^{-1}\left(\frac{x}{\sqrt{x^2 + ln(K)}}\right)\right] \times \left[exp\left(-\frac{x^2}{sin^2\left(\frac{\pi}{4} + \frac{1}{2}sin^{-1}\frac{x}{\sqrt{x^2+ln(k)}}\right)}\right) + exp\left(-\frac{x^2}{2}\right)\right]. \quad (23)$$

where the right hand side is names as bound-2 and denoted by $b_2(x)$.

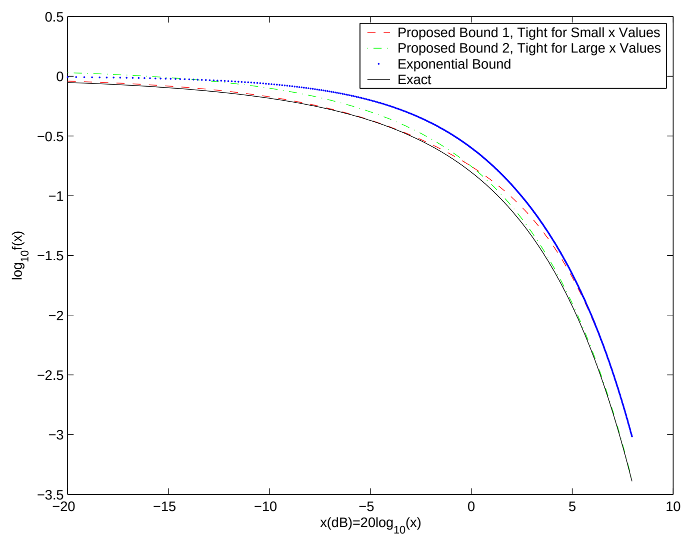

Fig. 5. Proposed Bounds ($K = 2$), Exponential Bound, and Exact Complementary Error Function

Computer simulation results for the proposed bounds with $K = 2$ are depicted in Fig. 5 where it is seen that the suggested bound-1 is tighter especially for low x values than the exponential bound given in ineq. (11). On the other hand bound-2 in ineq. (24) is better especially for large x values than the exponential bound in ineq. (11). As it is clear from Fig. 5 that the two bound can be combined to yield a better bound. These bounds can be combined into a single bound as using the unit step function $u(x)$ as:

$$b(x) = b_1(x) \times (u(x) - u(x-1)) + b_2(x) \times u(x-1) \quad (24)$$

which is written explicity as follows:

$$erfc(x) \leq \frac{2}{\pi}\left[(\frac{\pi}{4} - sin^{-1}(\sqrt{\frac{x^2}{x^2+K}}))\right.$$
$$\left. \times\ exp(-2x^2) + \frac{\pi}{4} \times exp(-x^2)\right] \times (u(x) - u(x-1))$$
$$+ \left[\frac{\pi}{4} - \frac{1}{2} \times sin^{-1}\left(\frac{x}{\sqrt{x^2+ln(K)}}\right)\right]$$
$$\times \left[exp\left(-\frac{x^2}{sin^2\left(\frac{\pi}{4}+\frac{1}{2}sin^{-1}\frac{x}{\sqrt{x^2+ln(k)}}\right)}\right)\right.$$
$$\left. + exp\left(-\frac{x^2}{2}\right)\right] \times (u(x-1)). \quad (25)$$

IV. Conclusions

In this manuscript we proposed new upper bounds for complementary error function. The proposed bounds are tighter when compared to the previously suggested exponential ones. The proposed bound approach the exact values in the limiting case.

References

[1] N. C. Beaulieu, "A simple series for personal computer computation of the error function $q(\cdot)$," *IEEE Trans. Commun.*, vol. 37, pp. 989–991, Sep. 1989.

[2] M. K. Simon and M. Alouini, "Exponential type bounds on the generalized marcum q function with application to error probability analysis over fading channels," *IEEE Trans. Commun.*, vol. 48, no. 4, pp. 359–366, Mar. 2000.

[3] M. Chiani, D. Dardari, and M. K. Simon, "New exponential bounds and approximations for the computation of error probability in fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 840–845, Jul. 2003.

[4] J. W. Craig, "A new, simple and exact result for calculating the probability of error for two-dimnetional signal constellations," in *IEEE MILCOM Conf. Rec.*, Boston, MA, 1991, pp. 2551–2555.

[5] M. K. Simon and D. Divsalar, "Some new twists to problems involving the gaussian probability integral," *IEEE Trans. Commun.*, vol. 46, pp. 200–210, Feb. 1998.